Leftist Number World

Initial problem: *We want to extend the natural numbers by a new element, that is larger than every natural number. What should this element look like?*

Problem 1. *Compute* $\ldots 999 + 1$.

Solution. *By intuitively using the natural extension of the usual addition algorithm we obtain*

$$\begin{array}{lr} & ...999999 \\ + & 1 \\ \hline & ...000000 \end{array}$$

We interpret this as $\ldots 999 + 1 = 0$.

Notation: $\overleftarrow{9} := \ldots 999$ or, in general

$$\overleftarrow{a_k \ldots a_1} b_n \ldots b_1 := \ldots a_k \ldots a_1 a_k \ldots a_1 b_n \ldots b_1.$$

These representations are called **leftist representations**. For instance, we say that $\overleftarrow{9}$ is the leftist representation of -1.

Problem 2. *(a) Compute* $\overleftarrow{9} + \overleftarrow{1}$. *Consider your result and think about whether it is a plausible result and why.*

(b) Compute $\overleftarrow{9} + \overleftarrow{2}$. *Consider your result and think about whether it is a plausible result and why.*

Solution. *(a) Again, we apply the natural extension of the addition algorithm and obtain*

$$\begin{array}{cccccc} & \ldots & 9 & 9 & 9 & 9 \\ + & \ldots & 1_1 & 1_1 & 1_1 & 1 \\ \hline & \ldots & 1 & 1 & 1 & 0 \end{array}$$

Using our new notation this reads $\overleftarrow{9} + \overleftarrow{1} = \overleftarrow{1}0$. *We are asked to assess the plausibility of our result. The following observation shows that it is consistent with the familiar rules of arithmetic. We have* $\overleftarrow{9} = 9 \cdot \overleftarrow{1}$, *so our calculation can be expressed as* $\overleftarrow{1}0 = 9 \cdot \overleftarrow{1} + 1 \cdot \overleftarrow{1} = 10 \cdot \overleftarrow{1}$, *which is quite plausible, as* $10 \cdot \overleftarrow{1} = \overleftarrow{1}0$ *is plausible.*

(b) We have

$$\begin{array}{cccccc} & \dots & 9 & 9 & 9 & 9 \\ + & \dots & 2_1 & 2_1 & 2_1 & 2 \\ \hline & \dots & 2 & 2 & 2 & 1 \end{array}$$

That is $\overleftarrow{9} + \overleftarrow{2} = \overleftarrow{2}\,1$*, and again we observe that this fits well with the familiar rules:*

$$\overleftarrow{2}\,1 = \overleftarrow{9} + \overleftarrow{2} = 9 \cdot \overleftarrow{1} + 2 \cdot \overleftarrow{1} = 11 \cdot \overleftarrow{1} = 10 \cdot \overleftarrow{1} + \overleftarrow{1} = \overleftarrow{1}\,0 + \overleftarrow{1}\,.$$

This is plausible, because

$$\begin{array}{cccccc} & \dots & 1 & 1 & 1 & 0 \\ + & \dots & 1_1 & 1_1 & 1_1 & 1 \\ \hline & \dots & 2 & 2 & 2 & 1 \end{array}$$

fits with our extended addition algorithm.

Problem 3. *Which number is larger:* $a = \overleftarrow{01}$ *or* $b = \overleftarrow{10}$*?*

Solution. *The solution is that we cannot answer this question in a simple manner. At least not by referring to the usual ordering of the natural numbers, namely the lexicographic order. This is because both numbers have no first digit and a comparison digit-by-digit gives alternating results. The last digit of* a *is larger than the last digit of* b*, the second last digit of* a *is smaller than the second last digit of* b*, and so on.*

Problem 4. *(a) Find the leftist representations of* -4 *and* -17*.*

(b) Does every negative integer have a leftist representation? If no, give a counterexample and explain why there can't be a leftist representation for your stated number. If yes, explain how to construct the leftist representation of a negative number in general.

Solution. *(a) By the above mentioned method we obtain*

$$\begin{array}{ccccc} & \dots & 0 & 0 & 0 \\ - & \dots & 0_1 & 0_1 & 4 \\ \hline & \dots & 9 & 9 & 6 \end{array}$$

Thus, $\overleftarrow{9}\,6$ *is the leftist representation of* -4*. Analogously we obtain that* $\overleftarrow{9}\,83 = -17$*, which we verify by adding for a change:*

$$\begin{array}{cccccc} & \dots & 9 & 9 & 8 & 3 \\ + & \dots & 0_1 & 0_1 & 1_1 & 7 \\ \hline & \dots & 0 & 0 & 0 & 0 \end{array}$$

(b) The results from (a) indicate that every negative integer has a leftist representation and (if necessary by considering further examples) we can observe a strong pattern. Informally one could describe the leftist representation of $-a$ for $a \in \mathbb{N}$ as follows: The representation is $\overleftarrow{9}$ followed by $10^n - a$, where n is the number of digits of a. As an example we determine the leftist representation of -267: We have $n = 3$, so we compute $10^3 - 267 = 733$, and the leftist representation of -267 is $\overleftarrow{9}\,733$.

We can also describe the leftist representation of a negative integer in a more formal manner. If a is an n-digit natural number, $10^n \cdot \overleftarrow{9} + (10^n - a)$ is the leftist representation of $-a$. The proof is straight forward:

$$10^n \cdot \overleftarrow{9} + (10^n - a) + a = 10^n \cdot \overleftarrow{9} + 10^n = 10^n(\overleftarrow{9} + 1) = 10^n \cdot 0 = 0.$$

Convention: We do not allow post decimal positions in leftist representations.

Problem 5. *(a) Does $\frac{1}{2}$ have a leftist representation? If yes, determine it. If no, explain why there cannot be such a representation.*

(b) Does $\frac{1}{3}$ have a leftist representation? If yes, determine it. If no, explain why there cannot be such a representation.

Solution. *(a) It is easily verified that $\frac{1}{2}$ has no leftist representation, because from $...a_3a_2a_1 = \frac{1}{2}$ it would follow that $2 \cdot \ldots a_3a_2a_1 = 1$. But that means that $2a_1$ would have to end on 1, which is impossible.*

(b) Let us assume we have a leftist representation $a = \ldots a_3a_2a_1$ of $\frac{1}{3}$, and see if this enables us to determine the digits of a. From $\frac{1}{3} = ...a_3a_2a_1$ it follows that $...a_3a_2a_1 \cdot 3 = 1$. Thus, $3a_1$ must end on 1, and we obtain $a_1 = 7$. As $3 \cdot 7 = 21$, we get a carry 2. Thus, $3a_2 + 2$ has to end on 0. Therefore, $3a_2$ has final digit 8, and we obtain $a_2 = 6$. From $3 \cdot 6 + 2 = 20$ we conclude that the next carry equals 2. Hence, $3a_3$ ends also on 8, and so on. This yields $6 = a_3 = a_4 = \ldots$ or, in other words, $\frac{1}{3} = \overleftarrow{6}\,7$.

*An **alternative solution** is the following: From $\overleftarrow{9} = -1$ it follows that $\overleftarrow{3} = -\frac{1}{3}$. As $\overleftarrow{6}\,7 + \overleftarrow{3} = 0$, we obtain that $\overleftarrow{6}\,7$ is the additive inverse of $-\frac{1}{3}$, that is $\overleftarrow{6}\,7 = \frac{1}{3}$.*

Problem 6. *Let n be a positive integer. Show that if $\frac{1}{n}$ has a leftist representation, then* $\gcd(n, 10) = 1$.

Solution. *Assume that $a = \ldots a_3a_2a_1$ is the leftist representation of $\frac{1}{n}$. It follows that $n \cdot \ldots a_3a_2a_1 = 1$. Thus, na_1 has final digit 1, or, in other words, $na_1 \equiv 1 \pmod{10}$, and therefore* $\gcd(n, 10) = 1$.

Problem 7. *Determine the leftist representation of $\frac{1}{7}$ or show that it does not exist.*

Solution. *We determine the representation of $\frac{1}{7}$ with the same technique that we used for determining the representation of $\frac{1}{3}$. Let $a = \ldots a_3a_2a_1 = \frac{1}{7}$. It follows that $7a_1 \equiv 1 \pmod{10}$. Thus, $a_1 = 3$. As $7 \cdot 3 = 21$ we have the carry 2. Hence, $7a_2 + 2 \equiv 0 \pmod{10}$, or, in other words $7a_2 \equiv 8 \pmod{10}$. Thus, $a_2 = 4$. We proceed in the same manner and obtain $\frac{1}{7} = \overleftarrow{285714}3$.*

Problem 8. *Compare the leftist representations of $-\frac{1}{3}$ and $-\frac{1}{7}$ with the decimal representations of $\frac{1}{3}$ and $\frac{1}{7}$. Formulate a conjecture, check it by considering another example, and justify it.*

Solution. *We have seen that the leftist representation of $-\frac{1}{3} = \overleftarrow{3}$ has the same digits as the decimal representation $0.\overline{3}$ of $\frac{1}{3}$. We have determined that $\overleftarrow{285714}3$ is the leftist representation of $\frac{1}{7}$. By computing $\overleftarrow{0} - \overleftarrow{285714}3$ (by using the subtraction algorithm or by doing mental arithmetic) we obtain that the leftist representation $-\frac{1}{7} = \overleftarrow{142857}$ has the same digits as the decimal representation $0.\overline{142857}$ of $\frac{1}{7}$. This leads to the following* ***conjecture***:

Let n be a positive integer. If $\frac{1}{n}$ has a decimal representation that becomes periodic just after the decimal point, say $\frac{1}{n} = 0.\overline{a_1a_2\ldots a_k}$, then $-\frac{1}{n}$ has a leftist representation and we have $-\frac{1}{n} = \overleftarrow{a_1a_2\ldots a_k}$.

The conjecture can easily be checked by means of another example. We only consider its proof here. Assume that $\frac{1}{n} = 0.\overline{a_1a_2\ldots a_k}$. It follows that $n \cdot 0.\overline{a_1a_2\ldots a_k} = 1 = 0.\overline{9}$. This implies $n \cdot \overleftarrow{a_1a_2\ldots a_k} = \overleftarrow{9} = -1$. Thus, we have established that $-\frac{1}{n} = \overleftarrow{a_1a_2\ldots a_k}$.

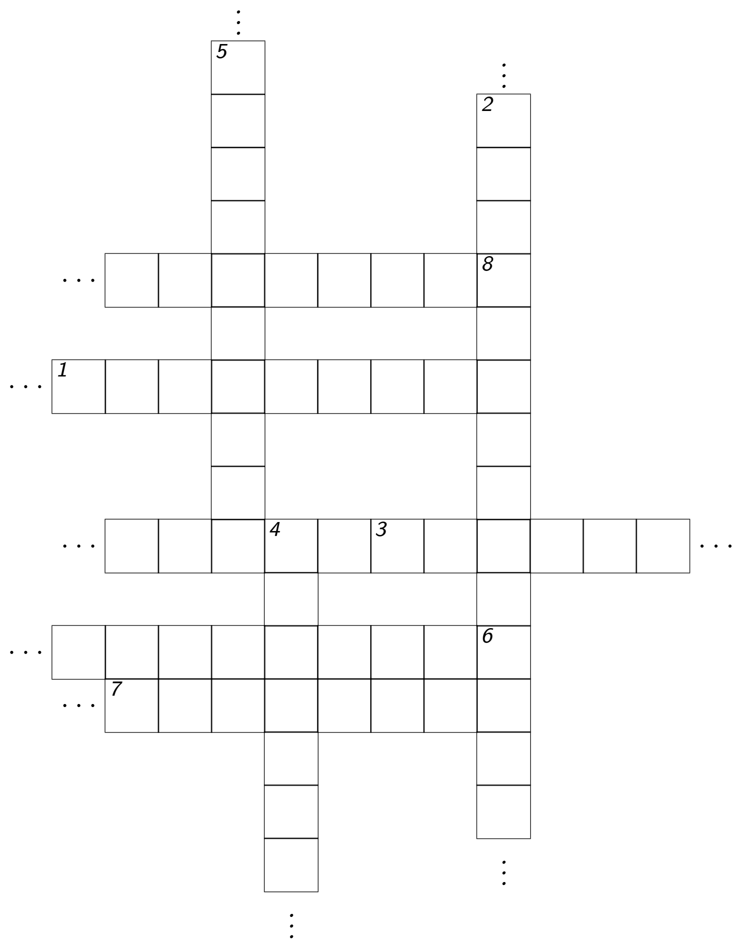
Problem 9. *Let n be a positive integer. Show that $\frac{1}{n}$ has a leftist representation if and only if $\gcd(10, n) = 1$.*

Solution. *We have already done the only-if part in Problem 6. Let us now assume that $\gcd(10, n) = 1$. Thus, $\frac{1}{n}$ has a decimal representation that becomes periodic just after the decimal point. By the solution of Problem 8 we know that this implies that $-\frac{1}{n}$ has a leftist representation (namely the one with the same digits), say $-\frac{1}{n} = \overleftarrow{a_1\ldots a_k}$. We let $b_i = 9 - a_i$ for $i = 1, \ldots, k-1$, and $b_k = 10 - a_k$. Then, $\overleftarrow{a_1\ldots a_k} + \overleftarrow{b_1\ldots b_k} = 0$, and therefore $\frac{1}{n} = -\left(-\frac{1}{n}\right) = \overleftarrow{b_1\ldots b_k}$.*

Problem 10. *Solve the cross-number puzzle by using the following clues.*

Across *1)* -27843 *3)* $-\frac{x}{9}$ *for some* $x \in \{0, 1, \ldots, 9\}$ *6)* $-p$ *for some prime number* p *7)* $-q$ *for some prime number* q*, whose square is between* 100 *and* 200 *8)* $2 \cdot \overleftarrow{9}$

Down *2)* $-\frac{3}{7}$ *4)* $-n$ *for some perfect number*[1] n *5)* -2^k *for some integer* $k > 3$



Solution. *The full solution can be seen below. Here are some brief explanations. It is necessary to fill in the solutions in a different order than the one presented here.*

(1) We have $-27843 = \overleftarrow{9}72157$ *by the solution of Problem 4 (b).*

(2) We have $\frac{3}{7} = 0.\overline{428571}$*. Analogously to the solution of Problem 8 we can see that this implies* $-\frac{3}{7} = \overleftarrow{428571}$

(3) As $\frac{x}{9} = 0.\overline{x}$*, we can conclude that* $-\frac{x}{9} = \overleftarrow{x}$ *for any digit* x*. The solution of (2) implies that the solution of (3) contains the digit* 2*. Thus,* $x = 2$ *and the solution equals* $\overleftarrow{2}$*.*

(4) The solutions of (6) and (7) imply that from the third last position on the digits of the leftist representation that we are looking for are all equal to 9*. Therefore, the decimal representation of the perfect number* n *has only two digits. The only perfect number*

[1] A perfect number is a positive integer that is equal to the sum of its positive divisors, excluding the number itself. For exampe, 6 is perfect, as $6 = 1 + 2 + 3$.

with two digits is 28*. Thus, we obtain* $-n = -28 = \overleftarrow{9}\,72$*. This goes together with the solution of (3).*

(5) The solution of (3) tells us that the decimal representation of 2^k *ends on* 8*. Moreover, the solution of (1) implies that this decimal representation has at most three digits. The only number of the form* 2^k *with* $k > 3$ *that meets these requirements is* 128*. Thus,* $-2^k = -128 = \overleftarrow{9}\,872$*.*

(6) The solution of (2) tells us that the prime p *we are looking for ends on* 5*. Thus,* $-p = -5 = \overleftarrow{9}\,5$*.*

(7) The solution of (2) tells us that the prime q *we are looking for ends on* 3*. As* $3^2 < 100$ *and* $23^2 > 200$*, the only possible choice is* $q = 13$*. Thus,* $-q = -13 = \overleftarrow{9}\,87$*.*

(8) We have $2 \cdot \overleftarrow{9} = 2 \cdot (-1) = -2 = \overleftarrow{9}\,8$*. This goes together with the solution of (2). Of course, the solution can also be determined by using the multiplication algorithm.*

				⋮									
				9									
				9					⋮				
				9					1				
				9					4				
				9					2				
	⋯	9	9	9	9	9	9	9	8				
				9					5				
⋯	9	9	9	9	7	2	1	5	7				
				8					1				
				7					4				
	⋯	2	2	2	2	2	2	2	2	2	2	2	⋯
					7				8				
⋯	9	9	9	9	9	9	9	9	5				
	⋯	9	9	9	9	9	9	8	7				
					9				1				
					9				4				
					9				⋮				
					⋮								

Further Explorations

Problem 11. *Compute* $\overleftarrow{3}\,7 : 3$.

Solution. *If we try to use the usual division algorithm, we encounter a problem, namely that we cannot start from the left as the dividend has no first digit. Thus, we have to think about how to start from the right. This means determining the last digit of the result first. As the dividend ends on* 7*, we know that* 3 *times the final digit of the result must end on* 7*. In other words, we look for a digit* a_1*, such that* $3a_1 \equiv 7 \pmod{10}$*. The unique solution is* $a_1 = 9$*. Now we have to subtract* $3 \cdot 9$ *from the dividend, that is* $\overleftarrow{3}\,7 - 27 = \overleftarrow{3}\,10$*. The result tells us that* 3 *times the second last digit of the result ends on* 1*, so the second last digit must be* 7*, and so on. We can put this in a scheme similar to the usual long division method:*

$$
\begin{array}{rl}
\ldots 3337 : 3 = & \ldots 7779 \\
27 & \\
\hline
\ldots 3331 & \\
21 & \\
\hline
\ldots 3331 & \\
21 & \\
\hline
\ldots 3331 & \\
21 &
\end{array}
$$

Hence, $\overleftarrow{3}\,7 : 3 = \overleftarrow{7}\,9$.

Problem 12. *Try other divisions with leftist dividends. Do you encounter a general problem? Describe it.*

Solution. *Further examples can be considered and the division algorithms in leftist notation and in usual decimal notation can be compared. One 'advantage' of the algorithm in leftist notation is that it needs no guessing ("How often goes this into that?"). In this method every digit of the result could be determined by table look-up. For more on the appeal of leftist representations for (computer) arithmetic see [**?**, **?**]. If long division in leftist notation is considered, one should be aware that this method only works for divisors that are relatively prime to* 10*. This is because the algorithm requires unique solutions in* $\{0, 1, \ldots, 9\}$ *for all congruences* $mx \equiv a \pmod{10}$*,* $a \in \{0, 1, \ldots, 9\}$*, where* m *is the divisor. These unique solutions exist if and only if* m *has an inverse element in* $\{0, 1, \ldots, 9\}$ *modulo* 10*. This is the case if and only if* $\gcd(m, 10) = 1$*. Note that we are not helpless if* $\gcd(m, 10) > 1$ *and we want to divide by* m*. For instance, if we want to divide by* 15*, we can just multiply both dividend and divisor by* 2*. Then we have to do a division by* 30*, which we can accomplish by dividing by* 3 *and adjusting the position of the decimal point afterwards.*

Problem 13. *Does $\sqrt{2}$ have a leftist representation? What about other square roots?*

Solution. *Assume that $a = \dots a_3 a_2 a_1$ is the leftist representation of $\sqrt{2}$. We try to determine the digits of a or to end up with a contradiction. As above, we first have to think about the final digit a_1. From $\sqrt{2} = \dots a_3 a_2 a_1$ it follows that $2 = (\dots a_3 a_2 a_1)^2$. Thus, if a_1 was even, that final digit of a^2 would have to be divisible by 4, which is not the case. But if a_1 was odd, the final digit of a^2 would have to be odd. This is a contradiction, so $\sqrt{2}$ has no leftist representation.*

Further Reading

The question which natural numbers have square roots that have a leftist representation can be further pursued, although it is pretty hard to establish an exhaustive answer. For more on this see (1). Other interesting activities are the hunt for zero divisors in the set of leftist numbers or for new solutions of the equation $x^2 = 1$. That both can be found in (1). There is much more to read about leftist numbers (the usual name is 10-adic numbers) and the (in some sense more important) p-adic numbers, where the base p is a prime. A good starting point are the following references.

(1) Carl, M., Schmitz, M. *What is worthy of investigation?*. Math Semesterber 69, 223–251 (2022). `https://doi.org/10.1007/s00591-022-00322-1`
Full text available at `https://arxiv.org/abs/2106.01408`

(2) Carl, M., Schmitz, M. *Discoveries in a 10-adic number world.* In: D. Sarikaya, K. Heuer, L. Baumanns, B. Rott (Eds.). “Problem Posing and Solving for Mathematically Gifted and Interested Students – Best Practices, Research and Enrichment”.

(3) A. Rich. *Leftist Numbers.* The College Mathematics Journal, Vol. 39, No. 5, pp. 330-336 (2008)