Leftist Number World

Initial problem: *We want to extend the natural numbers by a new element, that is larger than every natural number. What should this element look like?*

Problem 1. *Compute* $\ldots 999 + 1$.

Notation: $\overleftarrow{9} := \ldots 999$ or, in general

$$\overleftarrow{a_k \ldots a_1} b_n \ldots b_1 := \ldots a_k \ldots a_1 a_k \ldots a_1 b_n \ldots b_1.$$

These representations are called *leftist representations*. For instance, we say that $\overleftarrow{9}$ is the leftist representation of -1.

Problem 2. *(a) Compute* $\overleftarrow{9} + \overleftarrow{1}$. *Consider your result and think about whether it is a plausible result and why.*

(b) Compute $\overleftarrow{9} + \overleftarrow{2}$. *Consider your result and think about whether it is a plausible result and why.*

Problem 3. *Which number is larger:* $a = \overleftarrow{01}$ *or* $b = \overleftarrow{10}$*?*

Problem 4. *(a) Find the leftist representations of* -4 *and* -17.

(b) Does every negative integer have a leftist representation? If no, give a counterexample and explain why there can't be a leftist representation for your stated number. If yes, explain how to construct the leftist representation of a negative number in general.

Convention: We do not allow post decimal positions in leftist representations.

Problem 5. *(a) Does* $\frac{1}{2}$ *have a leftist representation? If yes, determine it. If no, explain why there cannot be such a representation.*

(b) Does $\frac{1}{3}$ *have a leftist representation? If yes, determine it. If no, explain why there cannot be such a representation.*

Problem 6. *Let* n *be a positive integer. Show that if* $\frac{1}{n}$ *has a leftist representation, then* $\gcd(n, 10) = 1$.

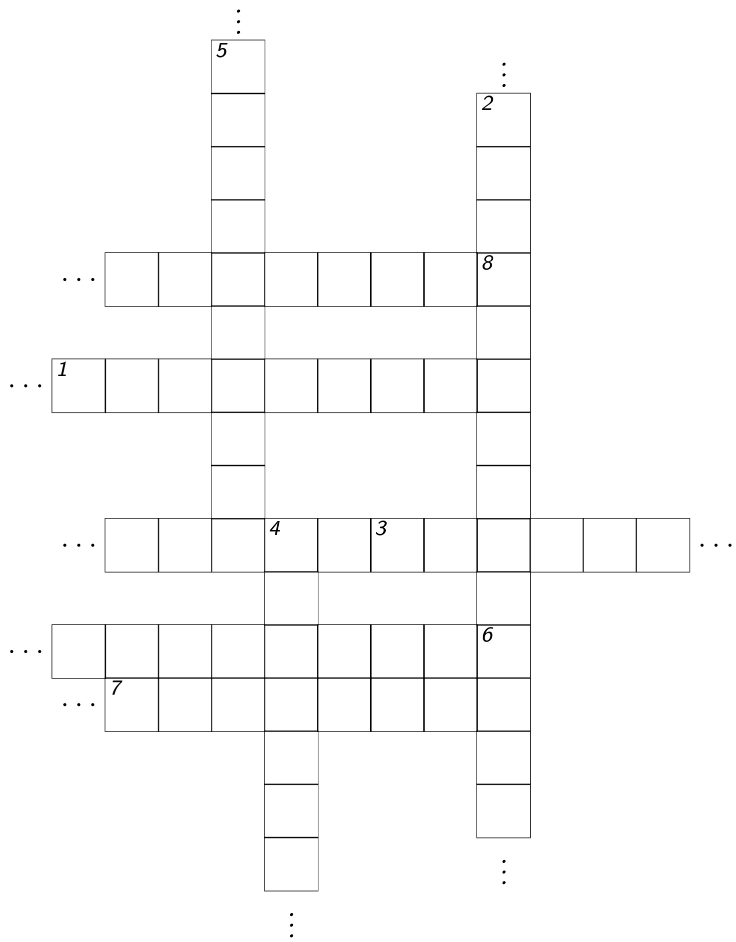
Problem 7. *Determine the leftist representation of $\frac{1}{7}$ or show that it does not exist.*

Problem 8. *Compare the leftist representations of $-\frac{1}{3}$ and $-\frac{1}{7}$ with the decimal representations of $\frac{1}{3}$ and $\frac{1}{7}$. Formulate a conjecture, check it by considering another example, and justify it.*

Problem 9. *Let n be a positive integer. Show that $\frac{1}{n}$ has a leftist representation if and only if* $\gcd(10, n) = 1$.

Problem 10. *Solve the cross-number puzzle by using the following clues.*

Across *1)* -27843 *3)* $-\frac{x}{9}$ *for some* $x \in \{0, 1, \ldots, 9\}$ *6)* $-p$ *for some prime number* p *7)* $-q$ *for some prime number* q*, whose square is between* 100 *and* 200 *8)* $2 \cdot \overleftarrow{9}$
Down *2)* $-\frac{3}{7}$ *4)* $-n$ *for some perfect number*[1] n *5)* -2^k *for some integer* $k > 3$



[1] A perfect number is a positive integer that is equal to the sum of its positive divisors, excluding the number itself. For exampe, 6 is perfect, as $6 = 1 + 2 + 3$.

Further Explorations

Problem 11. *Compute* $\overleftarrow{3}7 : 3$.

Problem 12. *Try other divisions with leftist dividends. Do you encounter a general problem? Describe it.*

Problem 13. *Does* $\sqrt{2}$ *have a leftist representation? What about other square roots?*

Further Reading

The question which natural numbers have square roots that have a leftist representation can be further pursued, although it is pretty hard to establish an exhaustive answer. For more on this see (1). Other interesting activities are the hunt for zero divisors in the set of leftist numbers or for new solutions of the equation $x^2 = 1$. That both can be found in (1). There is much more to read about leftist numbers (the usual name is 10-adic numbers) and the (in some sense more important) p-adic numbers, where the base p is a prime. A good starting point are the following references.

(1) Carl, M., Schmitz, M. *What is worthy of investigation?*. Math Semesterber 69, 223–251 (2022). `https://doi.org/10.1007/s00591-022-00322-1`
Full text available at `https://arxiv.org/abs/2106.01408`

(2) Carl, M., Schmitz, M. *Discoveries in a 10-adic number world.* In: D. Sarikaya, K. Heuer, L. Baumanns, B. Rott (Eds.). "Problem Posing and Solving for Mathematically Gifted and Interested Students – Best Practices, Research and Enrichment".

(3) A. Rich. *Leftist Numbers.* The College Mathematics Journal, Vol. 39, No. 5, pp. 330-336 (2008)